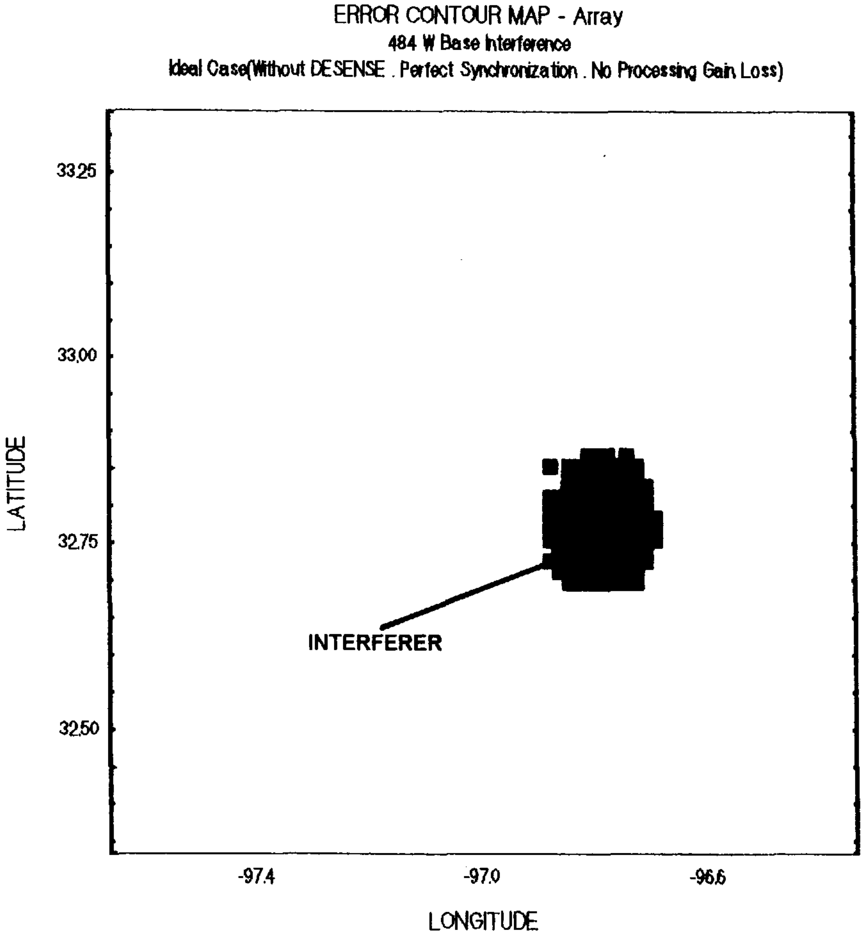


FIGURE 6 PinPoint Coverage in Presence of 484 W Interference

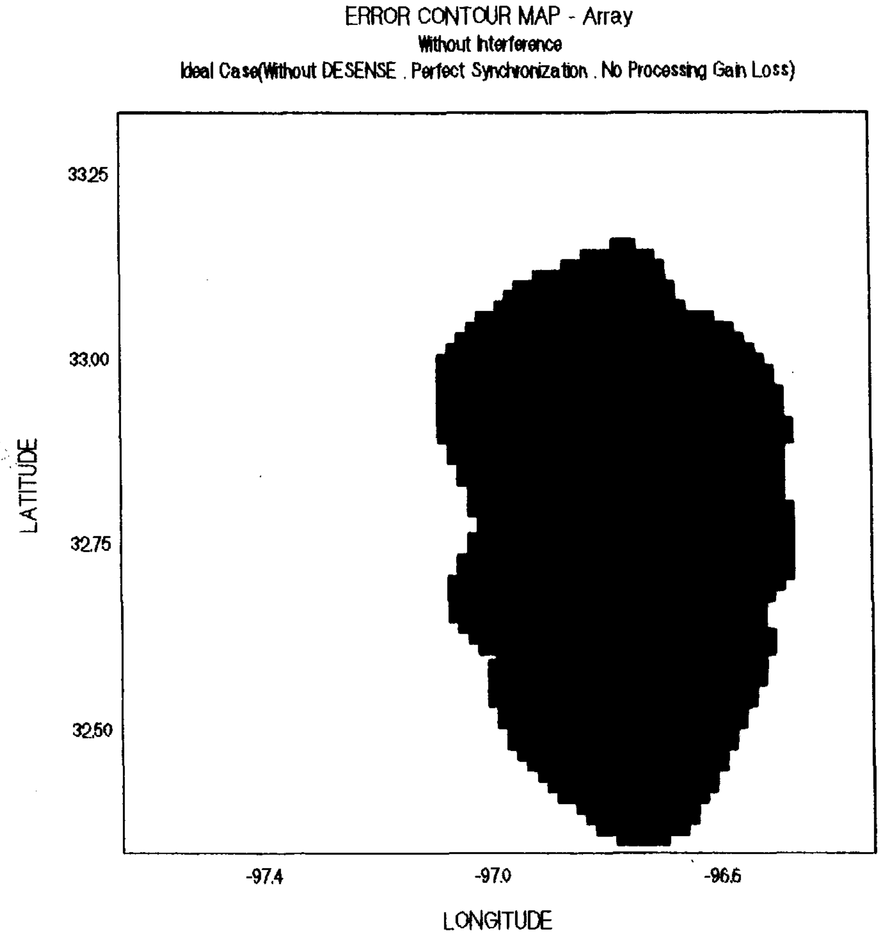


FIGURE 5 PinPoint Coverage without Interference

4 Experimental Validation of the Teletrac Model

To match the model predictions with real-world system performance, extensive experiments were conducted on the operating Teletrac LMS system in Dallas-Fort

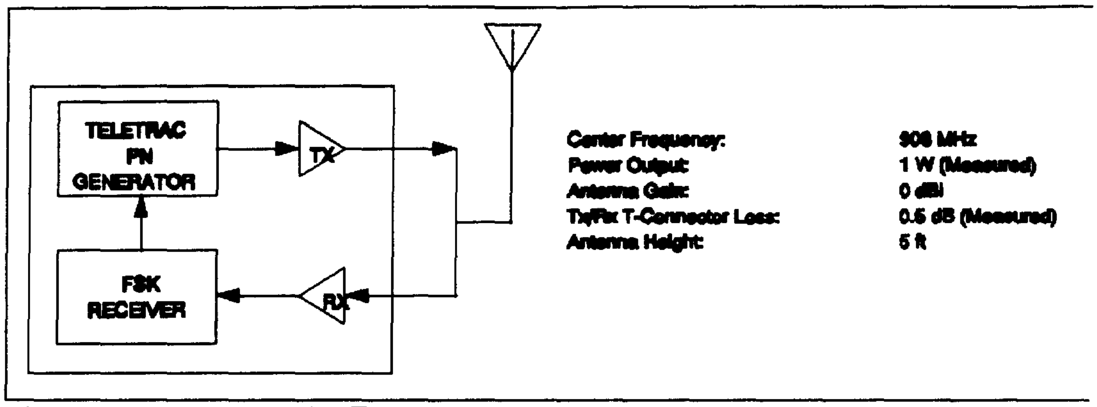


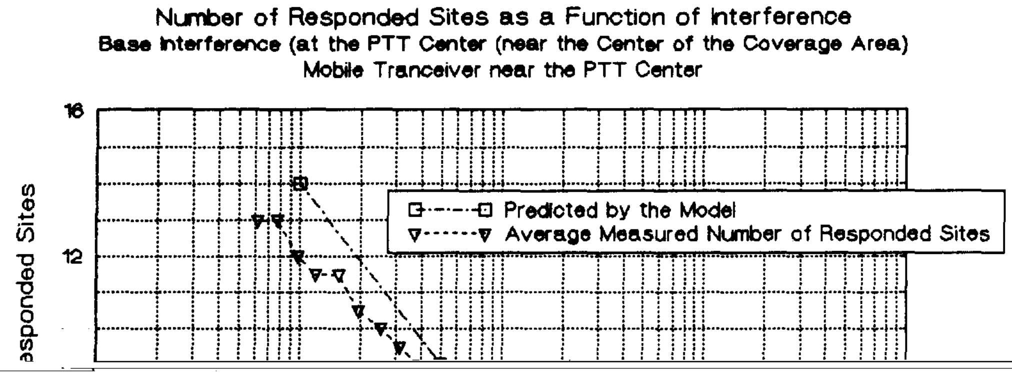
Figure 8 Teletrac Mobile Transceiver

Attenuation (dB)	Interferer ERP (watts)	Number of Sites
2	38.9	1
3	30.9	1
4	24.5	1
5	19.5	1
6	15.5	1
7	12.3	1
8	9.8	2
9	7.8	2
10	6.2	2
11	4.9	2
12	3.9	3
13	3.1	3
14	2.5	4
15	2.0	4

16	1.6	5
17	1.2	4 to 6
18	0.980	5 to 7
19	0.780	6 to 7
20	0.620	5 to 10
21	0.490	8 to 10
22	0.390	8 to 10
23	0.310	9 to 10
24	0.250	9 to 11
25	0.195	9 to 12
26	0.155	10 to 13
27	0.120	11 to 12
28	0.098	11 to 13
29	0.078	12 to 14
30	0.062	12 to 14

Table 2 Effect of Line-of-sight Interference

Figure 9 plots the average number of sites detecting a pulse as a function of the



Number of Responded Sites as a Function of Interference
Base Interference (at the PTT Center (near the Center of the Coverage Area)
Mobile Tranceiver near the PTT Center
16
12
esponded Sites
Predicted by the Model
Average Measured Number of Responded Sites

b) with interfering transmitter L1 activated (interference from a forward link).

Attribute	Value
Location	320 ft AGL (32.7628,-96.63463)
Center Frequency	908 Mhz
Chip Rate	1.7 Mcps
Effective Radiated Power	32.36 watts

Table 3 Characteristics of the Interference Source

4.2.2 Effect of Interference from a Forward-link Transmitter

Figure 10 plots both the predicted coverage area and the locations where the system performs with reasonable accuracy in the absence of added interference (test a). We here define reasonable accuracy as a DRMS of 500 feet or less. One sees immediately that the measured coverage area is extensive and corresponds well to the predictions of the model.

Figure 11 plots similar predicted and measured data in the presence of the simulated forward link interference. As one can see coverage falls sharply and there is again reasonable agreement between the measured data and the predictions. Table 4 below displays the agreement between the model and the measurements. Because the model does not include topographic data, variations of the scale shown in the table are to be expected.

	Model Prediction	Experimental Result	Percentage Agreement
Points in the Coverage Area	26	19	73%
Points Outside the Coverage Area	22	18	82%

Table 4 Model Match with Experiment

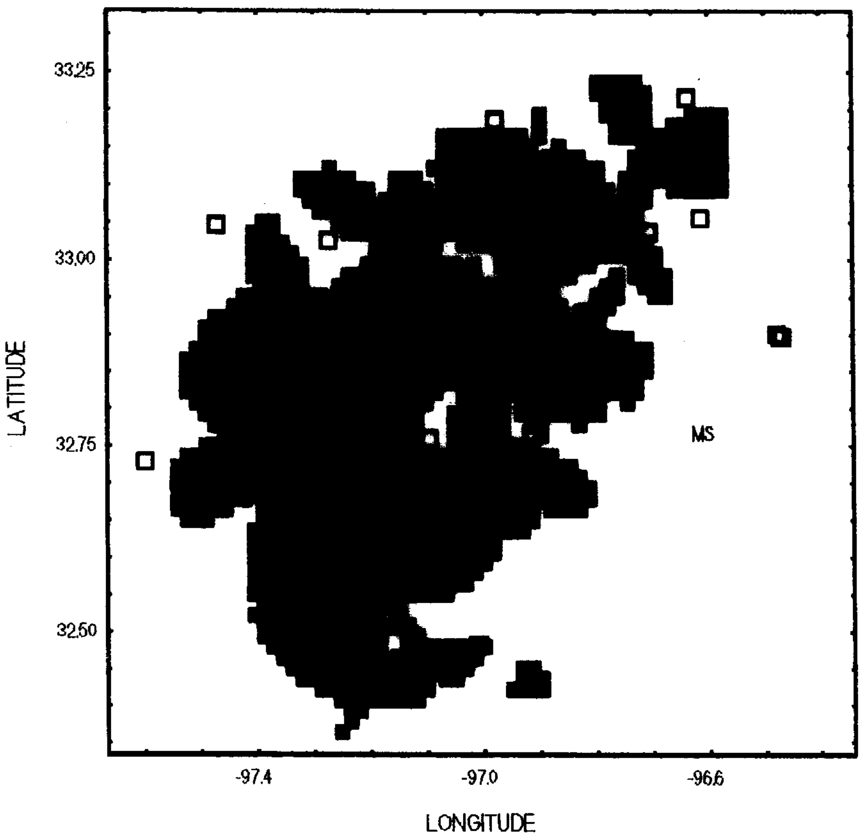


FIGURE 11. Measurements Points in Presence of 32 W Interference

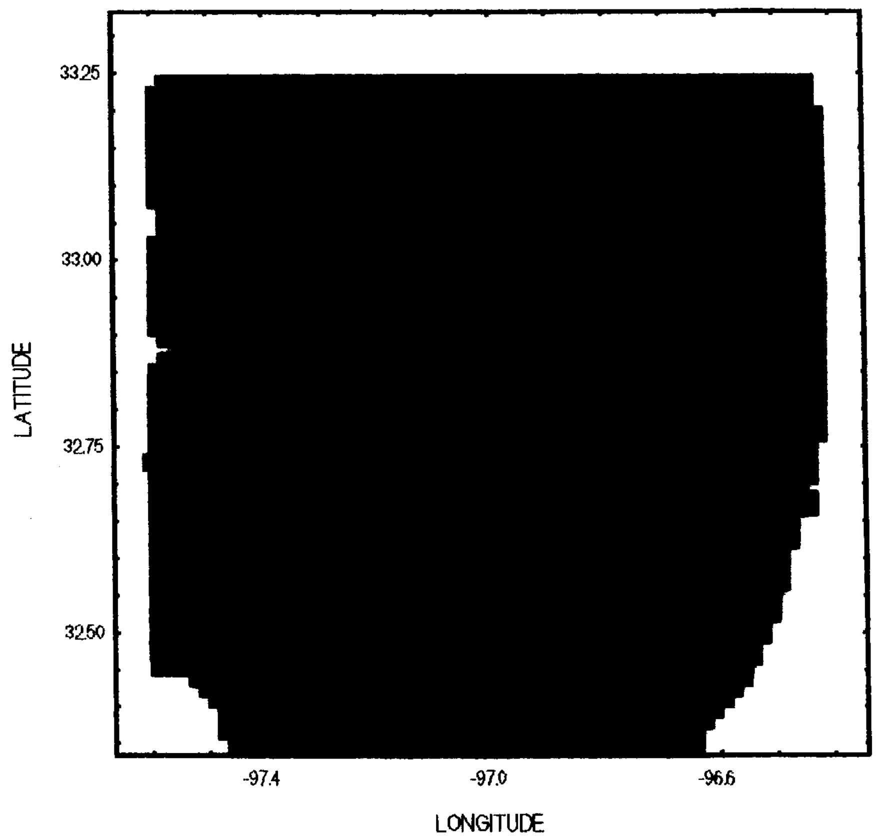


FIGURE 12. Measurements Points Without Interference

5 Performance of the Teletrac Receiver

We measured the variance of the time-of-arrival measurements made by the receivers in the Teletrac system as follows:

- Noise was set at a fixed level of -80 dBm.
- An RLU was connected to an attenuator. The output of the attenuator was connected to a long coaxial cable. This cable permitted physical separation of the transmitter and receiver to avoid errors caused by any

The receiver performance tracks the Cramér-Rao bound. Notice that the system performs well (rms error of 20 ns) at a signal-to-noise ratio of -20 dB!

6 Conclusions

The Teletrac model matches well with measurements made on Teletrac's Dallas-Fort Worth system. The model predicted that the coverage area would be substantially degraded in the presence of interference from a co-channel forward link of another LMS system. Measurements confirmed this prediction. Applying the Teletrac model to an LMS system with parameters similar to those of Pinpoint's proposed system shows that performance of such a system would be profoundly degraded in the presence of co-channel forward link transmissions from a similar system.

References

[1] E. Wildauer, "Impact of Interference on the Accuracy of Hyperbolic Location," PacTel Teletrac Technical Report, June 1993.

[2] IEEE Transactions on Vehicular Technology, Vol. 37, No. 1, Feb. 1988

[3] Pinpoint License Application for Dallas (2/9/93), file number 347485.

A

APPENDIX A

Impact of Wide-band Co-channel Interference on the Accuracy of Hyperbolic Location

June 22, 1993

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Abstract

The impact of wide-band interference on the location accuracy of the hyperbolic (TDOA) location solution is considered.

Review of a complete solution for the TDOA location and a model for wide-band interference and its impact on the location are presented. An n-Percentile error computation was performed for several theoretical and practical examples.

1 INTRODUCTION

In recent years ground-based location systems operating in the ISM band (902 -928 MHz) under part 90 provision, have been licensed and built (PacTel Teletrac), licensed (METS) or petitioned for the license (Pinpoint, [1]). Virtually all systems employ hyperbolic multilateration. The purpose of this work is to investigate the impact of sharing the spectrum (spread spectrum receivers using the same band) on the location accuracy of the hyperbolic multilateration (TDOA) systems.

Hyperbolic location systems, often called time difference of arrival (TDOA) systems, locate a transmitter through processing time of arrival measurements at four or more receive sites. The measurements from those receive sites are sent to a master receive site which computes the time differences and solves the location.

Declassified in the early 70s, spread spectrum techniques significantly improved the accuracy of TOA measurement. Still, the accuracy of ranging (TOA) is a function (S/N) of the received signal. This work will investigate the impact of signal to interference ratio (S/I) on the accuracy of the measurement and subsequently on the accuracy of the location.

To assess the lower bound of the location accuracy, we will consider the interference to be the **only** source of the errors in the TOA measurement. Under assumption of AWGN (which of course is valid in the case when the bandwidth of the interference is equal to or wider than the one for the signal), the interference will be modeled as a white noise with the variance according to Cramer-Rao lower bound.

The accuracy of hyperbolic systems was considered by several authors [2,3]. With the assumption that the measurement error has a zero mean and a Gaussian distribution, we will present an LMS solution for the hyperbolic multilateration.

We will derive a crude but reasonable measure of the location accuracy for a given probability - n-Percentile error (in another words, probability that the solution vector is contained by the ellipse of solutions (error ellipse)).

Section 2 will present the location solution for an hyperbolic system.

The propagation model and (S/I) computation will lead us to the variance of TOA measurement error in Section 3. Section 4 will review the statistics of the location solution distribution and derive a measure for the solution accuracy.

We will present some theoretical and practical results in Section 5. A brief discussion of the results and conclusions will be presented in Section 6.

2 HYPERBOLIC MULTILATERATION

2.1 Basic Equations, TOA Measurements and TDOA

The solution is general. For simplicity we will restrict equations to two-dimensional position location in a plane.

Let us assume that we performed TOA measurements at N receive sites.

Then we have:

$$
\begin{aligned}
t_1 &= t_0 + d_1 c + e_1 \\
t_2 &= t_0 + d_2 c + e_2 \\
&\vdots \\
t_N &= t_0 + d_N c + e_N
\end{aligned}
\qquad \text{(A1)}
$$

where:

- t_j - the measured TOA of the signal at jth receive site
- t_0 - the time (unknown) at which the signal was transmitted
- d_j - the distance from the jth receive site to the unknown position of the transmitter
- e_j - an error which accounts for TOA errors due to receiver noise (interference), errors in receive site position, propagation anomalies (multipath) and, in general, possible unknown bias due to somewhat unsynchronized clocks
- c - speed of light

Throughout this work we will assume (see Introduction and Section 2) that the only source of the errors is wide-band interference. Since we are modeling the interference as AWGN, the errors can be modeled as uncorrelated zero-mean random variables. Thus if e denotes the vector of errors e_j,

$$
e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}
$$

then $E[e] = 0$, where E denotes expectation. Moreover, since errors are uncorrelated, the covariance matrix R_e takes the form:

$$R_e = E\left[ee^T\right] = \begin{bmatrix} \sigma_{t_1}^2 & & & 0 \\ & \sigma_{t_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{t_N}^2 \end{bmatrix} \qquad \text{(A2)}$$

where $\sigma_{t_j}^2$ are variance of TOA measurements (see Section 3).

If we denote the vector of the unknown position of the transmitter by X:

$X = [\,x, y\,]^T$, denote the reference point (guess) by $X_0 = [\,x_0, y_0\,]^T$ and

denote the position of jth receive site by $S^j = \left[x^j, y^j\right]$, then we can write:

$$d_j = \left\|X - S^j\right\| = \left[\left(x - x^j\right)^2 + \left(y - y^j\right)^2\right]^{\frac{1}{2}}$$

and

$$d_{0_j} = \left\|X_0 - S^j\right\| = \left[\left(x_0 - x^j\right)^2 + \left(y_0 - y^j\right)^2\right]^{\frac{1}{2}}$$

where d_j and d_{0_j} are the distances from the jth receive site to unknown and reference position of the transmitter, respectively.

Expanding d into Taylor series in the vicinity of X_0 and keeping only the terms below second order we have after substituting into (A1):

$$1/c*\left[d_{0_j} + \frac{\partial d_j}{\partial x}\Big|_{x = x_0}\left(x_o - x^j\right) + \frac{\partial d_j}{\partial y}\Big|_{y = y_0}\left(y_0 - y^j\right)\right] = \left(t_j - t_0\right) + e_j \qquad \text{(A3)}$$

$$j = 1, \ldots, N$$

If we denote the following (NxN) matrix by F:

$$F = \begin{bmatrix} \frac{\partial d_1}{\partial x} & \frac{\partial d_1}{\partial y} \\ \frac{\partial d_2}{\partial x} & \frac{\partial d_2}{\partial y} \\ \vdots & \vdots \\ \frac{\partial d_N}{\partial x} & \frac{\partial d_N}{\partial y} \end{bmatrix}_{X=X_0} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \\ \vdots & \vdots \\ \cos\theta_N & \sin\theta_N \end{bmatrix} \tag{A4}$$

$$\theta_j = \tan^{-1}\left[\frac{y^j - y_0}{x^j - x_0}\right]$$

then we can write:

$$d_0 - F[X - X_0] = c*[(t - t_0) - e] \tag{A5}$$

where $t = [t_1, t_2, \ldots, t_N]^T$

$$t_0 = [t_0, t_0, \ldots, t_0]^T$$

We will eliminate t_0 from Equation A5 by subtracting each equation from its predecessor. The resulting system of N - 1 equations can be written in the following matrix form:

$$H(t + d_0/c) = H[F(X - X_0)]/c + n \tag{A6}$$

where H is an (N - 1) by N matrix:

$$H = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

and

$$n = He \tag{A7}$$

Consequently, the covariance matrix for n can be expressed:

$$R_n = E\{(He)(He)^T\} = HR_eH^T \tag{A8}$$

Equation A6 is the basic hyperbolic (TDOA) equation, and Equation a8 gives the covariance error matrix of the TDOA.

2.2 LMS Position Estimator and Covariance Matrix of Errors of the Estimator

If measurements are Gaussian and zero mean unbiased (which we assumed before), the optimal estimator of the position X which satisfies (6) is the least-mean squares (LMS) estimator (Markov theorem [4]).

The least-mean estimator is an estimator which minimizes the following quadratic form (Actually exponent of the Gaussian distribution of the errors):

$$P = n^T R_n^{-1} n \qquad \text{(A9)}$$

where n can be expressed from (A6)

$$n = H[t + (d_0 - F(X - X_0))/c)] \qquad \text{(A10)}$$

To determine the necessary condition for the minimum we will take the gradient of (A9), equate it to zero, then solve it for X:

The solution will give us:

$$\hat{X} = X_0 + c\left(F^T H^T R_n^{-1} HF\right)^{-1} F^T H^T R_n^{-1}(Ht - Hd_0/c) \qquad \text{(A11)}$$

where $d_0 = [d_{01}, d_{02}, \ldots, d_{0N}]^T$ - vector of distances from reference position to receive sites, F is given by (4) and R_n is given by (A8).

If we denote by Q the covariance matrix of errors in the X estimation, then:

$$\begin{aligned} Q &= E\left[\left(\hat{X} - E(\hat{X})\right)\left(\hat{X} - E(\hat{X})\right)^T\right] \\ &= c^2\left(F^T H^T R_n^{-1} HF\right)^{-1} \\ &= c^2\left(F^T H^T\left(HR_e H^T\right)^{-1} HF\right)^{-1} \end{aligned} \qquad \text{(A12)}$$

where $E(\hat{X}) = X_0$

To assess the accuracy of the LMS solution we must compute Q as a function of the transmitter and receive site position together with interference induced errors.

2.3 Computation of Covariance matrix of Errors

Let us denote:

$$L = (1/c^2) F^T H^T (HR_e H^T)^{-1} HF \qquad \text{(A13)}$$

L is an inverse of the covariance matrix and is positive definite, symmetric NxN matrix, and R_e is a diagonal matrix given by (A2).

It can be shown ([5]), that:

$$H^T(HR_eH^T)^{-1}H = [I - MU(U^TMU)^{-1}U^T] * M[I - U(U^TMU)^{-1}U^TM]$$

where U^T is an N-dimensional vector of 1's: [1,1,...,1] and $M = R_e^{-1}$.

We will rewrite M in the form:

$$M = \frac{c^2}{(\sigma^* c)^2}\begin{bmatrix} m_1 & & 0 \\ & m_2 & \\ 0 & & \ddots \\ & & & m_N \end{bmatrix}$$

where $m_j \triangleq \left(\frac{\sigma^*}{\sigma_{t_j}}\right)^2$

$(\sigma^* c)^2$ is mean-squared ranging error:

$$(\sigma^* c)^2 = \frac{1}{N}\sum_{i=1}^{N}(c \cdot \sigma_{t_i})^2$$

Then we can write:

$$L = K^TMK, \qquad \text{(A14)}$$

where K is given by:

$$K = I - U(U^TMU)^{-1}U^TMF \qquad \text{(A15)}$$

(K is NxN matrix)

$$K = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \\ 0 & & & \ddots \\ & & & & 1 \end{bmatrix} - \frac{1}{\sum_{j=1}^{N} m_j}\begin{bmatrix} m_1 & m_2 & \cdots & m_N \\ m_1 & m_2 & \cdots & m_N \\ \vdots & \vdots & & \vdots \\ m_1 & m_2 & \cdots & m_N \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} v_1 - \overline{v_1} \\ v_2 - \overline{v_2} \\ \vdots \\ v_N - \overline{v_N} \end{bmatrix}$$

(A16)

where $v_j \triangleq [\cos\theta_j \quad \sin\theta_j]$ is a jth row of matrix F.

and

$$\bar{v}_j \triangleq \frac{\sum_{k=1}^{N} m_k v_k}{\sum_{k=1}^{N} m_k}$$

Substituting (A16) into (A15) and (A14) yields:

$$L = \frac{1}{(\sigma^{*} c)^2} \begin{bmatrix} \sum_{j=1}^{N} m_j G_{x_j}^2 & \sum_{j=1}^{N} m_j G_{x_j} G_{y_j} \\ \sum_{j=1}^{N} m_j G_{x_j} G_{y_j} & \sum_{j=1}^{N} m_j G_{y_j}^2 \end{bmatrix} \quad \text{(A17)}$$

where,

$$G_{x_j} \triangleq \cos\theta_j - \frac{\sum_{k=1}^{N} m_k \cos\theta_k}{\sum_{k=1}^{N} m_k}, \quad G_{y_j} \triangleq \sin\theta_j - \frac{\sum_{k=1}^{N} m_k \sin\theta_k}{\sum_{k=1}^{N} m_k}$$

Matrix L is an inverse covariance matrix.

Finally, the covariance matrix of errors for LMS solutions, Q, is:

$$Q = \begin{bmatrix} \sigma_x^2 & \rho_{xy} \\ \rho_{xy} & \sigma_y^2 \end{bmatrix} \quad \text{(A18)}$$

where σ_x^2, σ_y^2 and ρ_{xy} are given by:

$$\sigma_x^2 = \left(\sigma^* c\right)^2 \frac{\sum_{j=1}^{N} m_j G_{y_j}^2}{\sum_{j=1}^{N} m_j G_{x_j}^2 \cdot \sum_{j=1}^{N} m_j G_{y_j}^2 - \left(\sum_{j=1}^{N} m_j G_{x_j} G_{y_j}\right)^2}$$

$$\sigma_y^2 = \left(\sigma^* c\right)^2 \frac{\sum_{j=1}^{N} m_j G_{x_j}^2}{\sum_{j=1}^{N} m_j G_{x_j}^2 \cdot \sum_{j=1}^{N} m_j G_{y_j}^2 - \left(\sum_{j=1}^{N} m_j G_{x_j} G_{y_j}\right)^2}$$

$$\rho_{xy} = -\left(\sigma^* c\right)^2 \frac{\sum_{j=1}^{N} m_j G_{x_j} G_{y_j}}{\sum_{j=1}^{N} m_j G_{x_j}^2 \cdot \sum_{j=1}^{N} m_j G_{y_j}^2 - \left(\sum_{j=1}^{N} m_j G_{x_j} G_{y_j}\right)^2}$$

Note: N is the number of sites in the line of sight that actually received the signal.

2.4 Covariance Matrix and GDOP

Geometric dilution of precision (GDOP) is a key position error mechanism. This error mechanism arises when the multilateration geometry of the measurement receive sites' positions generates lines of position which are nearly collinear. When such a condition exists, the errors (bias, position uncertainty of the receive sites, multipath and, as in our case, interference) can be blown up by mutual geometry of the receive sites, represented by GDOP.

GDOP is defined as the ratio of the root-mean-square position error to the root-mean-square ranging error.

It can be shown that given the covariance matrix of errors Q,

$$GDOP = \sqrt{trace\ [Q]} / c\sigma^* \qquad \text{(A19)}$$

where the denominator was determined in the previous paragraph.

Then we can write:

$$GDOP = \frac{1}{c\sigma^*}\sqrt{\sigma_x^2 + \sigma_y^2} \qquad \text{(A20)}$$

where σ_x^2 and σ_y^2 are elements of the main diagonal of Q.

Bad GDOP can be very damaging for the performance of the location system. Best GDOP will be achieved when receive sites are equally spaced in a circle around the area of interest.

2.5 TOA, Location Accuracy and Interference

In Section 3 we will show that TOA at the jth receiver is proportional to $(I_j)^{1/2}$, where I_j is interference power at the jth receiver.

In the case in which the interferer is at an approximately equal distance from all receivers and the received interferer power is proportional to its transmitted power, it follows from the equations of the previous chapters that the location accuracy is also proportional to the interference power at the source.

$$\text{Hyperbolic Error} \sim \sqrt{I} \tag{A21}$$

when the interferer is at an approximately equal distance from all receivers.

3 ESTIMATION OF THE TOA VARIANCE AS A FUNCTION OF INTERFERENCE

3.1 Cramer-Rao Bound for the Variance of Measured TOA

Under the assumption that interference is AWGN, the Cramer-Rao bound for a time of arrival estimate gives [2]:

$$\sigma_t^2 \geq \left[(2E/N_0)\beta_r^2\right]^{-1} \tag{A22}$$

where E is the energy in the received signal, $N_0/2$ is the two-sided noise power spectral density, and β_r^2 is a function of the bandwidth of the signal. If $S(\omega)$ denotes the Fourier transform of the signal, then

$$\beta_r = \left[\frac{\int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |S(\omega)|^2 \, d\omega}\right]^{1/2} \tag{A23}$$

is the "effective bandwidth" introduced by Gabor.

Let us denote $\beta_r = aR$, where R is the chip rate, and a is given by Table 1. Then, since $N_0 = N/B$ and $E = S \cdot T$, where T is the total length of the signal and B is the bandwidth of the receiver:

$$\sigma_t^2 \geq \frac{T_c}{2a^2 RB(S/N)T} \tag{A24}$$

Modulation	a ($\beta_r = aR$)
BPSK	2
Phase Shaped BPSK	1.875
MSK	1.45

Table 1

Equation A24 is the theoretical lower bound for variance of the time of arrival estimate.

For BPSK this formula reduces to the more familiar:

$$\sigma_t^2 \geq \frac{T_c}{4B^2(S/N)T} \tag{A25}$$

3.2 Propagation Model and (S/I) Computation

In general, the propagation loss is due to free space loss and additional urban loss. The free space loss is applicable to the case in which there is line of sight.

3.2.1 Free Space Loss

Free space loss given by the following formula (for the jth site):

$$L^j_{free} = 20\lg(f_{MHz}) + 20\lg d_j + 32.45 \quad (dB) \tag{A26}$$

Here:

L^j_{free} denotes free space loss (dB)

f_{MHz} - frequency in MHz

d_j - distance between source and jth receive site (km)

($d_j \leq 1$ km)

3.2.2 Urban Loss

We will use the Okumura model [6] for additional urban loss.

"This model is based on measurements made in Tokyo and suburbs. Statistical analysis of measurements was used to determine distance and frequency dependence of median field field strength, location variability and antenna height gain factors. The urban curves with suburban correction factors seem to be most suited for cities in the U.S." [6]

Figures 1 and 2 depict additional loss according to Okumura for different heights of the source and 6 ft height of the mobile. The relevant curves are labeled "SUBURBAN".

We will denote *UL(H)* additional urban loss between the source at height 6 ft (mobile) and jth receive site at height *H* ft.

If the interferer will be at a significant height (> 50 m), the only source for its losses will be the free space loss.

3.2.3 (S/I) at the Input of the jth Receiver

Assuming that the wide-band interferer is at a significant height (as in the case of the Pinpoint base station[1]) and the mobile height is 6 ft, we have for the jth receiver located at height H:

$$\widetilde{(S/I)}^j = 10\lg(P_T/P_I) - 20\lg(d_{j_s}/d_{j_I}) - UL_s^j(H) + G_T - G_I \quad \text{(A27)}$$

Here:

$\widetilde{(S/I)}^j$ - is S/I in dB at the input of the jth receiver.

P_T - mobile transmitted power (W)

P_I - interferer power (W)

d_{j_s} and d_{j_I} - distance between jth receiver mobile and interferer, respectively (km)

$UL_s^j(H)$ - urban loss for the mobile, receiver at the height H (dB)

G_T - antenna gain of the mobile (dB)

G_I - antenna gain of the interference source (dB)

Then:

$$(S/I)^j = 10^{\frac{\widetilde{(S/I)}^j}{10}} \quad \text{(A28)}$$

is the signal to interference ratio we will use in computation of TOA variance at the jth receiver, which is given by:

$$\sigma_{t_j}^2 \geq \frac{1}{2a^2 RB(S/I)^j 10^{\frac{PG}{10}}} \quad \text{(A29)}$$

where B - is bandwidth of the receiver (Hz)
R - is the chip rate
a - is given by Table 1
PG - receiver processing gain (dB)

4 STATISTICS OF THE LOCATION ACCURACY: n-PERCENTILE ERROR (P_e - ERROR)

4.1 Estimator Accuracy

We assumed before that our measurements have Gaussian distribution. Then we can write the probability density function for the errors in the hyperbolic location estimate - X, as:

$$f_x(z) = \left[2\pi|Q|^{1/2}\right]^{-1} \exp\left[-(1/2)(z - E(X))^T Q^{-1}(z - E(X))\right] \qquad \text{(A30)}$$

where Q is given by (A18):

The loci of constant density function (A30) values are given by:

$$(z - E(X))^T Q^{-1}(z - E(X)) = k \qquad \text{(A31)}$$

For an unbiased estimator at any given position we have (after solving the position):

$$E(X) = 0$$

So, we can write the following equation for the 2-dimensional envelope which encloses the values of X, which have probability of appearing smaller than, say P_e.

$$z^T Q^{-1} z = k \qquad \text{(A32)}$$

According to [2], on a plane this surface is an ellipse, and the following equation holds:

$$P_e(k) = 1 - exp(-k/2) \qquad \text{(A33)}$$

The semimajor and semiminor axis of this ellipse is given by $(k\lambda_1)^{1/2}$ and $(k\lambda_2)^{1/2}$, respectively.

Here λ_1 and λ_2 are eigenvalues of Q^{-1}.

Since the estimator is unbiased, and we are looking for the largest error value for a certain P_e, the following equation will give us an estimation for the bound of P_e-error:

$$\Delta(P_e) = \sqrt{(-2\ln(1 - P_e)\lambda_1)} \qquad \text{(A34)}$$

Here, Δ is an error, and λ_1 is given by: